

ON THE STRUCTURE OF EQUATIONALLY COMPLETE VARIETIES. II

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ABSTRACT. Each member \mathcal{V} of a large family of nonassociative or, when applicable, nondistributive varieties has the following universal property: Every variety \mathcal{K} that satisfies certain very weak versions of the amalgamation and joint embedding properties is isomorphic, as a category, to a coreflective subcategory of some equationally complete subvariety of \mathcal{V} . Moreover, the functor which serves to establish the isomorphism preserves injections. As a corollary one obtains the existence of equationally complete subvarieties of \mathcal{V} that fail to have the amalgamation property and fail to be residually small. The family of varieties universal in the above sense includes commutative groupoids, bisemigroups (i.e., algebras with two independent associative operations), and quasi-groups.

This paper continues the investigations begun in [15]. We study equationally complete varieties by treating them as categories in the usual way and comparing their structure by categorical means with the structure of arbitrary, noncomplete varieties. Our plan is to look at the assortment of varieties that can be represented as subcategories of complete varieties of various kinds. We want the representation to preserve a large number of algebraic properties so that we can infer as much information as possible about the structure of complete varieties from the structure of the varieties representable in them. In this regard our approach differs substantially from the main body of current work on representation problems of this nature. Cf. for example, Hedrlin and Lambek [6], Hedrlin and Pultr [7], and Sichler [16]. In particular we do not require the representation to be a full subcategory. On the other hand, in order to get any interesting results of the type we seek it does seem necessary that the representing functor have an inverse of some kind, as well as preserve injectivity at various levels. These considerations lead us in [15] to the following definition.

A functor \mathfrak{F} from the variety \mathcal{K} to the variety \mathcal{L} is called *polyinjective* if it satisfies the following three conditions. (All unexplained terminology from category theory comes from Mac Lane [9].)

(1) \mathfrak{F} is injective both as an object and arrow function, i.e., it is an isomorphism between \mathcal{K} and a subcategory of \mathcal{L} .

(2) \mathfrak{F} as an arrow function preserves injections, i.e., for all $\mathfrak{A}, \mathfrak{B} \in \mathcal{K}$ and every homomorphism $h: \mathfrak{A} \rightarrow \mathfrak{B}$, the homomorphism $\mathfrak{F}h: \mathfrak{F}\mathfrak{A} \rightarrow \mathfrak{F}\mathfrak{B}$ is one-one whenever h is.

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(3) There exists a forgetful functor \mathcal{G} from \mathcal{L} into \mathcal{K} and a natural transformation η from the identity functor \mathcal{I} on \mathcal{K} into $\mathcal{G} \circ \mathcal{I}$ such that $\eta_{\mathfrak{A}}: \mathfrak{A} \rightarrow \mathcal{G}\mathfrak{A}$ is an injection for every $\mathfrak{A} \in \mathcal{K}$.

In [15] it is shown that, if no a priori restrictions are placed on \mathcal{L} , then any \mathcal{K} that contains a null object (i.e., each of its members has a unique one-element subalgebra) can be represented by a polyinjective functor in a complete \mathcal{L} . It is also shown that, if we restrict our attention to locally finite \mathcal{L} , then every regular variety which is generated by a finite algebra can be represented in this way. These results allow us in [15] to exhibit complete varieties, even locally finite ones, that fail to have the amalgamation property. This provides a negative answer to a question of Fajtlowicz [5]. They are also used in [15] to construct a complete, locally finite variety that fails to be residually small, thus solving a problem of Clark and Krauss [4].

In the present paper we restrict ourselves to subvarieties \mathcal{L} of a fixed variety \mathcal{V} . The complete varieties of most of the more familiar algebras have been classified, and they turn out to have a rather regular structure. Taken together with the fact that almost nothing was known about the structure of complete varieties in general this led naturally to questions such as those of Fajtlowicz, Clark, and Krauss. It turns out however that for a comprehensive class of \mathcal{V} the categorial method described above can be used to exhibit complete subvarieties with very complex structure. The \mathcal{V} we deal with are the so-called *normal universal* varieties introduced in [14]. Their exact extent is not known, but they seem to include a large number of different nonassociative or, when it makes sense, nondistributive varieties of algebras. In particular, groupoids, commutative groupoids, semigroups with a unary operation adjoined, and bisemigroups are all shown in [14] to be normal universal. (By a *bisemigroup* we mean an algebra with two independent associative operations.) It is shown in [13] that quasi-groups have many of the characteristic properties of normal universal varieties. In the main result of the present paper we prove that each \mathcal{K} of a certain kind can be represented in a complete subvariety \mathcal{L} of any given normal universal \mathcal{V} . The condition \mathcal{K} must satisfy for this result to apply is much different from the existence of a null object, but it is weak enough to assure that the structure of the complete subvarieties of \mathcal{V} can be almost as complex as arbitrary varieties.

1. Statement of theorem. For \mathcal{G} to be a *forgetful* functor we mean that it commutes with the underlying set functor, i.e., \mathfrak{A} and $\mathcal{G}\mathfrak{A}$ always have the same universe, and, for any homomorphism $h: \mathfrak{A} \rightarrow \mathfrak{B}$, $\mathcal{G}h$ coincides with h . (This is a somewhat more general notion of forgetful functor than considered in [15].) Any appropriate system $\rho_0, \rho_1, \rho_2, \dots$ of terms, or polynomial symbols, from the language of \mathcal{L} gives rise to a forgetful functor \mathfrak{X}_{ρ} : for every $\mathfrak{B} \in \mathcal{L}$ take

$$\mathfrak{X}_{\rho}\mathfrak{B} = \langle B, \rho_0^{\mathfrak{B}}, \rho_1^{\mathfrak{B}}, \dots \rangle$$

where $\rho_i^{\mathfrak{B}}$ is the polynomial operation of \mathfrak{B} naturally associated with ρ_i ; we call $\mathfrak{X}_{\rho}\mathfrak{B}$ the ρ -transform of \mathfrak{B} . It is a forgetful functor from \mathcal{L} into any variety \mathcal{K} which includes $\mathfrak{X}_{\rho}(\mathcal{L})$. Conversely, it is well known from work of Mal'cev [10] that

every forgetful functor is of the form \mathfrak{Tr}_ρ for some definitional system ρ of terms. It is convenient for us to classify forgetful functors by their associated ρ .

It is also well known that every forgetful functor has a left adjoint. This can be easily obtained from the adjoint functor theorem. (See, for instance, Mac Lane [9, Theorem 2, p. 117].) We shall denote the left adjoint of \mathfrak{Tr}_ρ by \mathfrak{Ue}_ρ ; for each $\mathfrak{A} \in \mathcal{K}$ we call $\mathfrak{Ue}_\rho \mathfrak{A}$ the *universal ρ -envelope* of \mathfrak{A} in \mathcal{L} . The unit of adjunction \mathfrak{Tr}_ρ and \mathfrak{Ue}_ρ (see [9, p. 80 ff.]) is a natural transformation η from \mathfrak{S} to $\mathfrak{Tr}_\rho \circ \mathfrak{Ue}_\rho$ such that for each $\mathfrak{A} \in \mathcal{K}$ the homomorphism $\eta_{\mathfrak{A}}$ is a universal arrow from \mathfrak{A} to \mathfrak{Tr}_ρ .

In general \mathfrak{Ue}_ρ is not polyinjective. But this will be the case if \mathcal{L} is a subvariety of a normal universal \mathcal{V} , the definitional system ρ is properly chosen, and \mathcal{K} satisfies certain weak versions of the amalgamation and embedding property we now describe. For each $\mathfrak{A} \in \mathcal{K}$ let $\mathfrak{A}[X]$ be the free extension of \mathfrak{A} in \mathcal{K} by the free set of generators X . \mathcal{K} is said to have the *flat amalgamation property* if, for every $\mathfrak{A} \in \mathcal{K}$ and every subalgebra \mathfrak{B} of \mathfrak{A} , there exist a $\mathfrak{C} \in \mathcal{K}$ and injections $h: \mathfrak{A} \rightarrow \mathfrak{C}$ and $g: \mathfrak{B}[X] \rightarrow \mathfrak{C}$ such that $hb = gb$ for every b in \mathfrak{B} . This definition is due to Bacsich [1]; he shows among other things that for varieties the flat amalgamation property is equivalent to the *equational interpolation property*; see Jónsson [8, Corollary 4.6].

With the above definition in mind we shall say that \mathcal{K} has the *flat embedding property* if, for every $\mathfrak{A} \in \mathcal{K}$ and free algebra \mathfrak{F} of \mathcal{K} , there exists a $\mathfrak{C} \in \mathcal{K}$ and injections $h: \mathfrak{A} \rightarrow \mathfrak{C}$ and $g: \mathfrak{F} \rightarrow \mathfrak{C}$.

We are now able to state the main result of the paper.

THEOREM 1.1. *Assume that \mathcal{V} is the variety of groupoids, commutative groupoids, semigroups with a unary operation adjoined, bisemigroups, or, more generally, any normal universal variety. Or assume \mathcal{V} is the variety of quasi-groups.*

Let \mathcal{K} be any variety with a countable number of fundamental operations that has both the flat amalgamation and flat embedding properties. Then there exists an equationally complete subvariety \mathcal{L} of \mathcal{V} and a forgetful functor \mathfrak{Tr}_ρ from \mathcal{L} to \mathcal{K} such that its left adjoint \mathfrak{Ue}_ρ is polyinjective. Moreover, if the theories of identities of both \mathcal{V} and \mathcal{K} are decidable, and \mathcal{K} has only a finite number of fundamental operations, then \mathcal{L} can be taken so that it also has a decidable theory.

Thus every variety \mathcal{K} with both the flat amalgamation and embedding properties is isomorphic to a coreflective subcategory of a complete subvariety of \mathcal{V} .

It turns out that the most common varieties have both the flat amalgamation and embedding properties. For instance, rings, commutative rings, rings with unit of a fixed characteristic, semigroups, monoids, the groups all have both properties. Observe that among these all but the last one fail to have the (full) amalgamation property, and all of them fail to be residually small. The observation gives

COROLLARY 1.2. *Assume \mathcal{V} is as in Theorem 1.1. Then \mathcal{V} includes an equationally complete subvariety that fails to have the amalgamation property and also fails to be residually small.²*

²Banachewski [2] appears to be the first one to systematically use adjoint functors in this way to transfer properties from one variety to another, in this case the property of having enough injectives.

The first example of a complete variety that fails to be residually small was given by Taylor [17].

2. Some preliminaries. Let \mathcal{K} be any variety, and I be the set of operation symbols which denote its fundamental operations in the formal equational language of \mathcal{K} . I is called the *language type*, or simply the *type*, of \mathcal{K} , and it is assumed to be countable. In fact every variety considered in this paper is automatically assumed to have only countably many fundamental operations.

The associated terms, or polynomial symbols, are called *I-terms*; they are constructed from the symbols of I and a fixed list of variable symbols x_0, x_1, x_2, \dots in the usual way. Operation symbols of positive rank are placed in front of their respective arguments in order to avoid having to use parentheses. Thus it always makes sense to speak of the *first operation symbol* of a term. The first three variables symbols x_0, x_1, x_2 in our list are denoted alternatively by x, y, z ; we use the lightface letters x and y as metavariables ranging over variable symbols.

The set of all *I-terms* is denoted by Te_I . Terms will be represented by lower case Greek letters, especially τ and σ . $|\tau|$ denotes the length of τ . Equations in the formal language of \mathcal{K} are written $\tau = \sigma$ where τ, σ are *I-terms*; Eq_I is the set of all *I-equations*. The *I-theory* of \mathcal{K} , in symbols $\Theta_I \mathcal{K}$, is the set of *I-equations* identically satisfied in every member of \mathcal{K} . For any $\Gamma \subseteq \text{Eq}_I$, $\Theta_I[\Gamma]$ denotes the *I-theory* axiomatized by Γ . More generally, $\Theta_J[\Gamma]$ is the *J-theory* axiomatized by Γ for any language type J which includes I , and $\text{Mo}_J \Gamma$ is the class of *J-algebras* which are models of Γ .

Let C be any set of constant symbols disjoint from I . A term of the expanded language type $I \cup C$ which does not contain any variable symbol is called an *I-word in C*, or simply a *word in C* if the language type I is understood. We denote the set of all *I-words in C* by $\text{Wd}_I(C)$. The lower case Roman letters w, v, u , and t (possibly with subscripts) are used exclusively to represent words. An $(I \cup C)$ -equation of the form $w = v$ where w and v are words is called a *relation in C*. Finally, a *presentation* is a 3-tuple $\langle C; \Delta; \mathcal{K} \rangle$ where C is a set of constant symbols, and Δ a set of relations in C . The members of C and Δ are respectively called the *generators* and *relations* of the presentation. The lower case Roman letters a, b , and c are used for generators in general, and x and y for free generators.

The algebra \mathfrak{A} presented by $\langle C; \Delta; \mathcal{K} \rangle$ is defined, up to isomorphism, by the usual universal property: \mathfrak{A} is in \mathcal{K} , is generated by C , and satisfies the relations of Δ . Moreover if \mathfrak{B} is any other member of \mathcal{K} which contains a set $\{\bar{c}: c \in C\}$ of elements satisfying Δ , then there exists a homomorphism $h: \mathfrak{A} \rightarrow \mathfrak{B}$ such that $hc = \bar{c}$ for every $c \in C$.

A useful method for studying the structure of a group defined by a particular presentation is to choose from among the words a transversal, that is a system of unique representatives of the elements of the group. This suggests the following

DEFINITION 2.1. Let $\langle C; \Delta; \mathcal{K} \rangle$ be a presentation and $\mathfrak{A} \in \mathcal{K}$ the algebra it presents. A function f from $\text{Wd}_I(C)$ into itself is called a **normal-form function** for $\langle C; \Delta; \mathcal{K} \rangle$ if it satisfies the following conditions for all $w, v \in \text{Wd}_I(C)$.

- (i) $w^{\mathfrak{A}} = v^{\mathfrak{A}}$ implies $fw = fv$,

$$(ii) (fw)^{\mathfrak{A}} = w^{\mathfrak{A}}.$$

It is called a **Schreier normal-form function** if, in addition,

$$(iii) fv = v \text{ whenever } v \text{ is a subword of } fw.$$

When the particular presentation for \mathfrak{A} is clear or of no consequence we shall refer to g as a *normal-form function* for \mathfrak{A} .

It is a well-known fact that condition (i) can be replaced by the following three purely combinatorial conditions; see for instance Mal'cev [11, p. 225 ff.]. (Mal'cev's result however is formulated for the more general situation where \mathcal{K} is a quasi-variety.)

(i') $fw_i = fv_i$ for $i < n$ implies $f(Qw_0 \dots w_{n-1}) = f(Qv_0 \dots v_{n-1})$ for all $Q \in I$ and $w_0, \dots, v_{n-1} \in \text{Wd}_I(C)$;

(i'') $fw = fv$ for all $w = v \in \Delta$;

(i''') for some set Ω of axioms of \mathcal{K} we have

$$f(\tau(w_0, \dots, w_{n-1})) = f(\sigma(w_0, \dots, w_{n-1}))$$

for every $\tau(x_0, \dots, x_{n-1}) = \sigma(x_0, \dots, x_{n-1}) \in \Omega$ and all $w_0, \dots, w_{n-1} \in \text{Wd}_I(C)$.

We shall see that the main part of the proof of Theorem 1.1 consists of constructing a normal-form function for the universal envelope $\mathbb{U}_p \mathfrak{A}$. We shall also see that the transform $\mathfrak{X}_p \mathbb{U}_p \mathfrak{A}$ is a free extension of \mathfrak{A} in \mathcal{K} and that the normal-form function for $\mathbb{U}_p \mathfrak{A}$ is most conveniently constructed starting with one for $\mathfrak{X}_p \mathbb{U}_p \mathfrak{A}$ having certain special properties. The existence of a function with such special properties is established in the next lemma.

From now on \mathfrak{A} is assumed to be a fixed but arbitrary algebra in \mathcal{K} . As is common practice in both model theory and combinatorial group theory we shall also treat the elements of the universe A of \mathfrak{A} as constant symbols which can represent themselves in a presentation of \mathfrak{A} . The (equational) diagram $\Delta_{\mathfrak{A}}$ of \mathfrak{A} is the set of all relations in the symbols A of the form $Qa_0 \dots a_{n-1} = a_n$ where $Q^{\mathfrak{A}}(a_0, \dots, a_{n-1}) = a_n$. Recall that $\mathfrak{A}[X]$ is the free extension of \mathfrak{A} in \mathcal{K} by the free generators X . Thus $\mathfrak{A}[X]$ has the presentation $\langle A \cup X; \Delta_{\mathfrak{A}}; \mathcal{K} \rangle$.

In the following lemma l represents a function that assigns a positive integer to each free generator in X ; l is arbitrary except for the requirement that $lx = 1$ for at least one $x \in X$. For any $w \in \text{Wd}_I(A \cup X)$, $|w|_l$ denotes the modified length of w obtained by assigning to each occurrence of any x the weight lx . More precisely, if $w = z_0 z_1 \dots z_{n-1}$ with $z_i \in I \cup A \cup X$,

$$|w|_l = \sum_{i < n} lz_i$$

where we extend l to the whole of $I \cup A \cup X$ by taking $lz = 1$ for $z \in I \cup A$.

LEMMA 2.2. *Let x_0 be a fixed but arbitrarily chosen element of X such that $lx_0 = 1$. There exists a Schreier normal-form function g for $\mathfrak{A}[X]$ (more precisely for $\langle A \cup X; \Delta_{\mathfrak{A}}; \mathcal{K} \rangle$) satisfying the following conditions for every $w \in \text{Wd}_I(A \cup X)$:*

(i) *for every $x \in X \sim \{x_0\}$, x occurs in gw only if it occurs in w ;*

(ii) $g(\text{Wd}_I(X)) \subseteq \text{Wd}_I(X)$;

(iii) $|gw|_l \leq |w|_l$ if $w \in \text{Wd}_I(X)$.

Furthermore, if \mathfrak{B} is a subalgebra of \mathfrak{A} given beforehand, and if B is its universe, then g can be taken to satisfy in addition

$$(iv) g(Wd_I(B \cup X)) \subseteq Wd_I(B \cup X).$$

PROOF. Let \mathfrak{B} be a fixed subalgebra of \mathfrak{A} . For each $w \in Wd_I(A \cup X)$ let $\phi w = \langle n, m, |w|_I \rangle$ where n and m are respectively the numbers of occurrences in w of generators from $A \sim B$ and B . Order these triples lexicographically.

Let α be any well-ordering of $I \cup A \cup X$ such that the first element of the ordering is x_0 . Define a well-ordering β of $Wd_I(A \cup X)$ as follows. For any $w, v \in Wd_I(A \cup X)$ we have $w <_\beta v$ (i.e. w precedes v under the ordering β) if either $\phi w < \phi v$, or $\phi w = \phi v$ and w precedes v under the lexicographic ordering of words induced by α (thinking here of words as sequences of symbols of $I \cup A \cup X$).³ Finally, for every $w \in Wd_I(A \cup X)$ take gw to be the first word v under the β -ordering such that $v^{\mathfrak{A}[X]} = w^{\mathfrak{A}[X]}$.

Clearly conditions (i) and (ii) of Definition 2.1 hold (with $g, \mathfrak{A}[X]$, and $Wd_I(A \cup X)$ in place of f, \mathfrak{A} , and $Wd_I(C)$, respectively). To check 2.1(iii) assume v is a subword of gw and $gv \neq v$. Then $gv <_\beta v$. Let u be the word obtained from gw by replacing v by gv . It is easy to check that $u <_\beta gw$. But this contradicts the defining condition of gw since clearly $u^{\mathfrak{A}[X]} = w^{\mathfrak{A}[X]}$. Therefore 2.1(iii) holds and g is a Schreier normal-form function for $\mathfrak{A}[X]$.

To see that g satisfies condition (i), suppose one of the free generators $x \in X \sim \{x_0\}$ occurs in gw but not in w . Let $(gw)(x/x_0)$ be the word obtained from gw by replacing each occurrence of x by x_0 . Let h be the endomorphism of $\mathfrak{A}[X]$ such that $ha = a$ for all $a \in A$, $hy = y$ for all $y \in X \sim \{x\}$, and $hx = x_0$. Since gw and w represent the same element of $\mathfrak{A}[X]$, and x does not occur in w , we have

$$((gw)(x/x_0))^{\mathfrak{A}[X]} = h((gw)^{\mathfrak{A}[X]}) = h(w^{\mathfrak{A}[X]}) = w^{\mathfrak{A}[X]}. \quad (1)$$

Also, since $lx_0 = 1 \leq lx$,

$$|(gw)(x/x_0)|_I \leq |(gw)|_I. \quad (2)$$

If strict inequality holds here we get $\phi((gw)(x/x_0)) < \phi(gw)$, and hence

$$(gw)(x/x_0) <_\beta gw. \quad (3)$$

If equality holds in (2), then $\phi((gw)(x/x_0)) = \phi(gw)$. But then again (3) holds since $x_0 <_\alpha x$, so $(gw)(x/x_0)$ precedes gw in the lexicographic ordering induced by α . However (1) and (3) together contradict the defining condition of gw . Thus (i) is satisfied.

To see that (ii) and (iii) are both satisfied let w be a word that does not contain any generator from A . Then $\phi w = \langle 0, 0, |w|_I \rangle$. Since $\phi(gw) < \phi w$, we must have $\phi(gw) = \langle 0, 0, |gw|_I \rangle$ and $|gw|_I \leq |w|_I$. This gives (ii) and (iii), and a similar argument gives (iv).

Suppose I is finite and \mathfrak{A} is finitely generated, or at least the free extension of a finitely generated algebra. Suppose further that the function l is recursive. Then, if $\mathfrak{A}[X]$ has a solvable word problem, the least word v under the β -ordering that

³We want to thank George Bergman who suggested the use of a lexicographic ordering of terms here; it considerably simplifies the proof.

represents the same element in $\mathfrak{A}[X]$ as w does can be effectively found. Thus under these circumstances g is recursive.

It is easy to see that the variety \mathcal{K} has the flat embedding property iff, for every $\mathfrak{A} \in \mathcal{K}$, the subalgebra of $\mathfrak{A}[X]$ generated by X is a free algebra of \mathcal{K} with X as a set of free generators. It is also easy to show that \mathcal{K} has the flat amalgamation property iff, for every algebra \mathfrak{A} of \mathcal{K} and subalgebra \mathfrak{B} of \mathfrak{A} , $\mathfrak{B}[X]$ is isomorphic to the subalgebra of $\mathfrak{A}[X]$ generated by $B \cup X$ where B is the universe of \mathfrak{B} . In view of these equivalences the following lemma is immediate.

LEMMA 2.3. *Let $\mathfrak{A} \in \mathcal{K}$ and g be a Schreier normal-form function for $\mathfrak{A}[X]$.*

(i) *If \mathcal{K} has the flat embedding property and g satisfies 2.2(ii), then g restricted to $\text{Wd}_I(X)$ becomes a Schreier normal-form function for the free algebra of \mathcal{K} generated by X .*

Let \mathfrak{B} be any subalgebra of \mathfrak{A} and let B be its universe.

(ii) *If \mathcal{K} has the flat amalgamation property and g satisfies condition 2.2(iv), then g restricted to $\text{Wd}_I(B \cup X)$ becomes a Schreier normal-form function for $\mathfrak{B}[X]$.*

Let J be another language type, and let $\rho: I \rightarrow \text{Te}_J$ be a function that assigns to each operation symbol $Q \in I$ of positive rank n a J -term, $\rho Q = \tau(x_0, \dots, x_{n-1})$; as usual writing a term τ in the form $\tau(x_0, \dots, x_{n-1})$ is meant to imply that the variable symbols that actually occur in τ all appear in the list x_0, \dots, x_{n-1} . If c is a constant symbol of I , i.e., a nullary operation symbol, then $\rho Q = \sigma(x_0)$. Any such ρ is called a *formal definition* of I in Te_J . As mentioned in §1 every formal definition ρ is associated with a forgetful functor \mathfrak{Xr}_ρ such that

$$\mathfrak{Xr}_\rho \mathfrak{B} = \langle B, (\rho Q)^{\mathfrak{B}} \rangle_{Q \in I}$$

for every J -algebra \mathfrak{B} . $\mathfrak{Xr}_\rho \mathfrak{B}$ is an I -algebra provided $(\rho Q)^{\mathfrak{B}}$ is a constant operation whenever Q is nullary.

DEFINITION 2.4. *Let ρ be a formal definition of I in Te_J . ρ is said to be **nonoverlapping** if the following conditions are satisfied for every $Q \in I$.*

(i) *If the rank n of Q is positive, then ρQ must contain at least one occurrence of each of the variables x_0, x_1, \dots, x_{n-1} .*

(ii) *Let $P \in I$ with rank m , and let $\sigma = \sigma(x_0, \dots, x_{m-1})$ be any subterm of ρP which does not consist of a single variable. Then for all J -terms $\tau_0, \dots, \tau_{n-1}, \pi_0, \dots, \pi_{m-1}$ we have*

$$\rho Q(\tau_0, \dots, \tau_{n-1}) \neq \sigma(\pi_0, \dots, \pi_{m-1})$$

unless $P = Q$ and $\sigma = \rho P$.

If ρ is nonoverlapping as a formal definition of I in Te_J , then it remains so when considered as a definition of I in Te_K for any language type K which includes J .

We shall need the following result due to McNulty [12, Theorem 2.9(v)].⁴

⁴See also S. Banach, *Sur un théorème de M. Sierpinski*, Fund. Math. **25** (1936), 5–6.

W. Sierpinski, *Sur les suites infinies de fonctions définies dans les ensembles quelconques*, Fund. Math. **24** (1935), 209–212.

J. Los, *Un théorème sur les superpositions des fonctions définies dans les ensembles arbitraires*, Fund. Math. **37** (1950), 84–86.

LEMMA 2.5. Assume J contains at least one operation symbol of rank 2 or more. Then there exists a nonoverlapping formal definition of I in Te_J .

Let ρ be a formal definition of I in Te_J , and let \mathcal{L} be a variety of type J . If $\mathfrak{Tr}_\rho(\mathcal{L}) \subseteq \mathcal{K}$, then \mathfrak{Tr}_ρ is a forgetful functor from \mathcal{L} into \mathcal{K} . Its left adjoint is denoted by $\mathfrak{Ue}_\rho^{\mathcal{L}}$, or simply \mathfrak{Ue}_ρ if \mathcal{L} is understood. It is clear that, for any $\mathfrak{A} \in \mathcal{K}$, $\mathfrak{Ue}_\rho^{\mathcal{L}}\mathfrak{A}$ is presented by $\langle A; \rho\Delta_{\mathfrak{A}}; \mathcal{L} \rangle$ where $\rho\Delta_{\mathfrak{A}}$ is the set of all relations of the form $\rho Q(a_0, \dots, a_{n-1}) = a_n$ where $Qa_0 \dots a_{n-1} = a_n \in \Delta_{\mathfrak{A}}$. If η is the unit of adjunction, then $\eta_{\mathfrak{A}}$ is the unique homomorphism from \mathfrak{A} into $\mathfrak{Tr}_\rho\mathfrak{Ue}_\rho^{\mathcal{L}}\mathfrak{A}$ that takes each $a \in A$ into the element of $\mathfrak{Ue}_\rho\mathfrak{A}$ denoted by a .

3. Definition of \mathcal{L} . Let \mathcal{K} be an arbitrary variety that has both the flat amalgamation and flat embedding properties, and let I be its language type. (We always assume I is countable.) In [15] an equationally complete variety \mathcal{L} which embeds \mathcal{K} is constructed by means of the following simple idea: Take $\mathcal{L}_0 = \mathcal{K}$. Let $\tau(x_0, \dots, x_{p-1}) = \sigma(x_0, \dots, x_{p-1})$ be an I -equation which is not in $\Theta_I\mathcal{K}$. Let R be a new operation symbol of rank 4, and let c_0, \dots, c_{p-1} be new constant symbols. Let Γ be the set of equations consisting of a set of axioms for $\Theta_I\mathcal{K}$ together with the two equations

$$R\tau(c_0, \dots, c_{p-1})\sigma(c_0, \dots, c_{p-1})yz = y; \quad Rxxyz = z;$$

in [15] this pair of equations is called a *local discriminating system* for $\tau = \sigma$. Take \mathcal{L}_1 to be the variety of language type $I \cup \{R, c_0, \dots, c_{p-1}\}$ axiomatized by Γ . Clearly \mathcal{L}_1 has no nontrivial subvariety satisfying the identity $\tau = \sigma$. Now choose an $(I \cup \{R, c_0, \dots, c_{p-1}\})$ -equation $\pi = \gamma$ not contained in the theory of \mathcal{L}_1 , and define a variety \mathcal{L}_2 by adjoining to the axioms of \mathcal{L}_1 a local discriminating system for $\pi = \gamma$. It is easy to see that by choosing the nonidentity at each level in a judicious way we obtain an infinite sequence $\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2, \dots$ of theories whose intersection $\mathcal{L} = \bigcap_{n < \omega} \mathcal{L}_n$, when properly defined to take into account the dissimilarity of language types, is minimal, i.e., equationally complete. Define $\mathfrak{Tr}_\rho\mathfrak{B}$ to be the I -reduct for every $\mathfrak{B} \in \mathcal{L}$. Then \mathfrak{Tr}_ρ is a forgetful functor from \mathcal{L} to \mathcal{K} . (The associated formal definition ρ is the identity: $\rho Q = Qx_0 \dots x_{m-1}$ for each $Q \in I$.) It turns out that \mathfrak{Ue}_ρ is polyinjective, although a more palpable polyinjective embedding of \mathcal{K} in \mathcal{L} is actually constructed in [15].

The complete variety \mathcal{L} whose construction is outlined in the preceding paragraph has an infinite language type no matter what the type of \mathcal{K} is. But for any given normal universal \mathcal{V} essentially the same construction will give a complete subvariety \mathcal{L} of \mathcal{V} with all the desired properties. In this situation the fundamental operations of \mathcal{K} , and the infinitely many new operations and constants used to form the local discriminating systems, must of course all be encoded into polynomials in the fundamental operations of \mathcal{V} . That there exist polynomials sufficiently independent of one another for this purpose is the essential characteristic of normal universal varieties.

In the proof of Theorem 1.1 we give below, instead of considering arbitrary normal-universal varieties, we confine our attention to the variety of groupoids. For this special case the argument is conceptually simpler and also essentially

independent of [14]. In the last section we shall indicate how the argument can be modified to give Theorem 1.1 in its full generality. From now on \mathcal{V} will always be the variety of groupoids, and \mathcal{L} will be a subvariety of \mathcal{V} . But \mathcal{K} will range over all varieties with a countable language type I . The unique binary operation symbol of \mathcal{V} is written \odot which we will write between its arguments. Thus for instance $(x \odot y) \odot z$ will stand for $\odot \odot xyz$.

We are now ready to define the complete variety \mathcal{L} of groupoids whose existence is asserted in Theorem 1.1. Let K be a denumerable set of operation symbols of rank 4 and L a denumerable set of constant symbols; we assume they are both disjoint from the type I of \mathcal{K} . Let J be the combined language type

$$J = I \cup K \cup L \cup \{\odot\}.$$

Take ρ to be a nonoverlapping formal definition of $I \cup K \cup L$ in Te_{\odot} ; such a definition exists by Lemma 2.5. In the sequel the restriction of ρ to I will also be denoted by ρ . Clearly for the purposes of proving the theorem we can identify \mathcal{V} with any variety definitionally equivalent to it (provided its theory is decidable when \mathcal{V} 's is), and it will be convenient to replace \mathcal{V} by its extension by the definitions of ρ . Hence we take

$$\mathcal{V} = \text{Mo}_J\{Qx_0 \cdots x_{m-1} = \rho Q : Q \in I \cup K \cup L\}.$$

We shall continue to refer to \mathcal{V} as the variety of groupoids, but observe that its language type is J . Since J contains \odot , ρ can be thought of as a formal definition of I in Te_J , and the associated forgetful functor \mathfrak{Tr}_{ρ} from \mathcal{V} to the variety of all algebras of type I coincides with the I -reduct. So for any $\mathfrak{B} = \langle B, Q^{\mathfrak{B}} \rangle_{Q \in J}$, $\mathfrak{Tr}_{\rho}\mathfrak{B} = \langle B, Q^{\mathfrak{B}} \rangle_{Q \in I}$.

We now define by recursion a transfinite, decreasing sequence $\mathcal{V} \supseteq \mathcal{L}_0 \supseteq \mathcal{L}_1 \supseteq \cdots \supseteq \mathcal{L}_{\omega} = \mathcal{L}$ of subvarieties of \mathcal{V} . Take \mathcal{L}_0 to be the class of all groupoids whose ρ -transform is contained in \mathcal{K} :

$$\mathcal{L}_0 = \mathcal{V} \cap \text{Mo}_J\Theta_I\mathcal{K}.$$

Observe that \mathcal{L}_0 is axiomatized by the set of equations $\Theta_I\mathcal{K} \cup \{Qx_0 \cdots x_{m-1} = \rho Q : Q \in I \cup K \cup L\}$. Take α to be a fixed ω -ordering of the J -equations. Assume \mathcal{L}_n has been defined for an arbitrary $n < \omega$. Let $\tau_n(x_0, \dots, x_{p-1}) = \sigma_n(x_0, \dots, x_{p-1})$ be the first J -equation under the ordering α such that $\tau_n = \sigma_n \notin \Theta_J\mathcal{L}_n$. (We shall show later that \mathcal{L}_n is nontrivial so that the equation always exists.) Choose distinct operation symbols $R_n \in K$ and $P_{n,0}, \dots, P_{n,p-1} \in L$ satisfying the following conditions.

- (1) $R_n \neq R_k$ and $P_{n,0}, \dots, P_{n,p-1} \neq P_{k,i}$ for all $k < n$ and all i .
- (2) R_n does not occur in $\tau_k = \sigma_k$ for any $k \leq n$.
- (3) $|\rho R_n|, |\rho P_{n,0}|, \dots, |\rho P_{n,p-1}| > |\tau_k(P_{k,0}, \dots, P_{k,l-1})|, |\sigma_k(P_{k,0}, \dots, P_{k,l-1})|$ for all $k \leq n$.

Observe that, since ρ is one-one and K and L are both infinite, the lengths of ρR for $R \in K$ must be unbounded, and similarly for the lengths of the ρP for $P \in L$. Thus conditions (1)–(3) can always be satisfied. Let

$$\Gamma_n = \{R_n\tau_n(P_{n,0}, \dots, P_{n,p-1})\sigma_n(P_{n,0}, \dots, P_{n,p-1})xy = x, R_nzzxy = y\}. \quad (4)$$

Take $\mathbb{L}_{n+1} = \mathbb{L}_n \cap \text{Mo}_J \Gamma_n$. Observe that

$$\mathbb{L}_n = \text{Mo}_J \left[\Theta_I \mathcal{K} \cup \bigcup_{k < n} \Gamma_n \cup \{ Qx_0 \cdots x_{m-1} = \rho Q : Q \in I \cup K \cup L \} \right]. \quad (5)$$

Finally we take $\mathbb{L} = \bigcap_{n < \omega} \mathbb{L}_n$ so that

$$\mathbb{L} = \text{Mo}_J \left[\Theta_I \mathcal{K} \cup \bigcup_{n < \omega} \Gamma_n \cup \{ Qx_0 \cdots x_{m-1} = \rho Q : Q \in I \cup K \cup L \} \right]. \quad (6)$$

This variety is clearly equationally complete provided it is not trivial. Its nontriviality will follow from the construction of the universal envelope in the next section.

4. Construction of the universal envelope. In the following \mathfrak{A} is a fixed but arbitrary member of \mathcal{K} . We shall construct a normal-form function for $\mathbb{L}_\rho \mathfrak{A}$ ($= \mathbb{L}_\rho^{\mathbb{L}} \mathfrak{A}$) or, more precisely, for its presentation $\langle A; \Delta_{\mathfrak{A}}; \mathbb{L} \rangle$. Our starting point is a normal-form function for \mathfrak{A} in \mathcal{K} , and, anticipating that this latter algebra will turn out to be a free extension of \mathfrak{A} in \mathcal{K} , we must start with a normal-form function for an appropriate $\mathfrak{A}[X]$.

Let X be a denumerable set of free generators disjoint from A , and let χ be a one-one correspondence between X and the subset of all words in $\text{Wd}_J(A)$ which begin with a symbol in $J \sim I (= K \cup L \cup \{\odot\})$. For any $w \in \text{Wd}_J(A \cup X)$ we shall denote by $w[\chi]$ the word in $\text{Wd}_J(A)$ obtained from w by replacing each $x \in X$ occurring in w by χx . Finally, choose a fixed element $x_0 \in X$ such that $|\chi x_0| = 1$. (Hence χx_0 must be one of the constant symbols of L .) Observe that $|w[\chi]| = |w|_I$ where I is the function that assigns $|\chi x|$ to x for each $x \in X$. Thus we can apply Lemma 2.2 to obtain a Schreier normal-form function for $\mathfrak{A}[X]$ satisfying conditions 2.2(i), (ii), (iv) and, in addition,

$$|(gw)[\chi]| \leq |w[\chi]| \quad \text{if } w \in \text{Wd}_J(X). \quad (1)$$

We now define a transfinite sequence $f_0, f_1, \dots, f_\omega = f$ of transformations of $\text{Wd}_J(A)$ by double recursion on the index of the transformation and on the length of the word $w \in \text{Wd}_J(A)$. We shall then prove that f_ω is a Schreier normal-form function for the presentation $\langle A; \Delta_{\mathfrak{A}}; \mathbb{L} \rangle$ of $\mathbb{L}_\rho \mathfrak{A}$. It will also be seen from this proof that each of the intermediate functions f_n , $n < \omega$, is a normal-form function for $\langle A; \Delta_{\mathfrak{A}}; \mathbb{L}_n \rangle$.

If $w \in A$ take $f_n w = gw = w$ for all $n \leq \omega$. Assume now that $|w| > 1$, or w is a constant symbol of J , and n is any natural number or ω . The definition of $f_n w$ separates into nine cases. It will be proved in the discussion following that the transformations are well defined. For simplicity we introduce the following abbreviations. (Recall the pair Γ_n of equations defined in (4) of §3.)

$$\tau_n(\bar{P}_n) = \tau_n(P_{n,0}, \dots, P_{n,p-1}), \quad \sigma_n(\bar{P}_n) = \sigma_n(P_{n,0}, \dots, P_{n,p-1}).$$

In the first four cases we assume that

$$w = Qv_0 \cdots v_{m-1}$$

where the first symbol Q of w (necessarily an operation symbol) is different from \odot , i.e., a member of $I \cup K \cup L$. Let $f_n^* w = Qf_n v_0 \cdots f_n v_{m-1}$. (This useful abbreviation and similar ones will be used repeatedly.) The various cases of the definition depend on the structure of $f_n^* w$.

Case 4.1. $Q \in I$. Let t be the word in $\text{Wd}_I(A \cup X)$ that is obtained from f_n^*w by taking each subword s that begins with a symbol in $J \sim I$, but is not itself properly included in another subword of f_n^*w of the form, and replacing it by the free generator $\chi^{-1}s$. It is easily seen that t is the unique I -word in the generators $A \cup X$ with the property that $f_n^*w = t[\chi]$; take $f_nw = (gt)[\chi]$.

Case 4.2. $Q = R_k$ with $k < n$ and

$$f_n^*w = R_k f_k(\tau_k(\bar{P}_k)) f_k(\sigma_k(\bar{P}_k)) ts.$$

Take $f_nw = t$.

Case 4.3. $Q = R_k$ with $k < n$ and $f_n^*w = R_k rrt$ s. Take $f_nw = s$.

Case 4.4. $Q = R_k$ with $k \geq n$, or $Q = P_k$ for any $k < \omega$, or $Q = R_k$ with $k < n$ but the hypotheses of Cases 4.2 and 4.3 both fail to hold. Take $f_nw = f_n^*w$.

For the remaining five cases we assume that w is of the form $v_0 \odot v_1$. In Cases 4.5 through 4.8, which respectively correspond to Cases 4.1 through 4.5, we assume that

$$f_n^*w = (f_nv_0) \odot (f_nv_1) = \rho Q(u_0, \dots, u_{m-1})$$

with $Q \in I \cup K \cup L$ and $u_0, \dots, u_{m-1} \in \text{Wd}_J(A)$. Since ρ is nonoverlapping, if f_n^*w has a representation of this form, it is unique.

Case 4.5. $Q \in I$. As in 4.1 there exists a unique $t \in \text{Wd}_I(A \cup X)$ such that $Qu_0 \dots u_{m-1} = t[\chi]$. Take $f_nw = (gt)[\chi]$.

Case 4.6. $Q = R_k$ with $k < n$ and $Qu_0 \dots u_{m-1} = R_k f_k(\tau_k(\bar{P}_k)) f_k(\sigma_k(\bar{P}_k)) ts$. Take $f_nw = t$.

Case 4.7. $Q = R_k$ with $k < n$ and $Qu_0 \dots u_{m-1} = R_k rrt$ s. Take $f_nw = s$.

Case 4.8. $Q = R_k$ and $k > n$, or $Q = P_k$, for any $k < \omega$, or $Q = R_k$ with $k < n$ but the hypotheses of Cases 4.6 and 4.7 both fail to hold. Take $f_nw = Qu_0 \dots u_{m-1}$.

Case 4.9. $f_n^*w \neq \rho Q(u_0, \dots, u_{m-1})$ for all $Q \in I \cup K \cup L$ and $u_0, \dots, u_{m-1} \in \text{Wd}_J(A)$. Take $f_nw = f_n^*w$.

Consider an arbitrary $n \leq \omega$. Observe that $R_i \neq R_j$ for distinct $i, j < \omega$ by (1) of §3, and recall that ρ is nonoverlapping by assumption. These observations show that each of the nine cases can individually be applied in only one way in a given situation, and two different cases cannot simultaneously apply with the possible exceptions of 4.2 and 4.3 and of 4.6 and 4.7. Thus f_0 is automatically well defined since none of these four latter cases can apply when $n = 0$, and in order to show f_n is also well defined for each positive $n \leq \omega$ it suffices to prove that

$$f_n(\tau_n(\bar{P}_n)) \neq f_n(\sigma_n(\bar{P}_n)) \quad \text{for all } n < \omega. \quad (2)$$

The proof of this nonequality requires a few preliminary facts. We first observe that the following holds for all $n \leq \omega$.

$$\text{If } w \in \text{Wd}_J(A) \text{ contains no generator in } A \text{ (i.e., } w \text{ is a constant } J\text{-term), then neither does } f_nw. \quad (3)$$

This is shown by an easy induction on the length of w . The only cases in the induction step that cause any problem are 4.1 and 4.5; we only need to consider the first one. Suppose w begins with a symbol $Q \in I$ and t is the unique word in

$\text{Wd}_I(A \cup X)$ such that

$$f_n^* w = t[\chi]. \quad (4)$$

Since w contains no A -generator, neither does $f_n^* w$ by the induction hypothesis. Thus t contains no A -generator, nor does χx for any $x \in X$ occurring in t . So by 2.2(ii), gt contains no A -generator, and by 2.2(i) every $x \neq x_0$ occurring in gt must occur in t . Hence $(gt)[\chi] = f_n w$ contains no generator in A . (Recall $\chi x_0 \in L$.)

We next prove that, for every $n \leq \omega$,

$$|f_n w| \leq |w| \text{ for every } w \in \text{Wd}_I(A) \text{ that contains no generator in } A. \quad (5)$$

This is also proved by induction on the length of w , and again the only nontrivial cases in the induction step are 4.1 and 4.5. Consider the first one. Then (4) holds with $t \in \text{Wd}_I(A \cup X)$. By (3), $f_n^* w$ contains no A -generator, so $t \in \text{Wd}_I(X)$. Thus combining (1) with the induction hypothesis we get

$$|f_n w| = |(gt)[\chi]| \leq |t[\chi]| = |f_n^* w| \leq |w|.$$

Case 4.5 is dealt with similarly. This establishes (5).

Consider now an arbitrary finite n , and let w be any subword of $\tau_n(\bar{P}_n)$ or $\sigma_n(\bar{P}_n)$. As an immediate consequence of (3) of §3 together with (5) we have

$$|f_n^* w| \leq |\tau_n(\bar{P}_n)| < |\rho P_{n,i}| \text{ for all } i.$$

Therefore, 4.8 with $Q = P_{n,i}$ for any i cannot apply in the definition of $f_n w$. This implies:

$$\begin{aligned} &\text{the relation } f_n w = f_n^* w \text{ is derivable from the defining identities} \\ &Qx_0 \dots x_{m-1} = \rho Q \text{ with } Q \in K \cup (L \sim \{P_{n,0}, \dots, P_{n,p-1}\}) \quad (6) \\ &\text{whenever 4.8 is applied with } w \text{ a subword of } \tau_n(\bar{P}_n) \text{ or } \sigma_n(\bar{P}_n). \end{aligned}$$

We make a second observation. Suppose again that w is a subword of $\tau_n(\bar{P}_n)$ or $\sigma_n(\bar{P}_n)$, and 4.1 applies in the definition of $f_n w$. Then by (3) we have $f_n^* w = t[\chi]$ where t contains no A -generator, i.e., $t \in \text{Wd}_I(X)$. Thus, since \mathcal{K} has the flat embedding property, we conclude by means of Lemma 2.3(i) that the relation $t = gt$ is identically satisfied in \mathcal{K} . Hence

$$\begin{aligned} f_n w = f_n^* w &\text{ is derivable from } \Theta_I \mathcal{K} \text{ whenever 4.1 is applied} \\ &\text{with } w \text{ a subword of } \tau_n(\bar{P}_n) \text{ or } \sigma_n(\bar{P}_n). \end{aligned} \quad (7)$$

A similar argument establishes the same result for 4.5 with $\Theta_I \mathcal{K}$ replaced by $\Theta_I \mathcal{K} \cup \{Qx_0 \dots x_{m-1} = \rho Q\}$ with $Q \in I$. Combining this with (6) and (7) we see by a simple inductive argument that for every subword w of $\tau_n(\bar{P}_n)$ or $\sigma_n(\bar{P}_n)$ the relation $f_n w = w$ is derivable from the identities

$$\begin{aligned} E = \Theta_I \mathcal{K} \cup \bigcup_{k < n} \Gamma_k \cup Qx_0 \dots x_{m-1} = \rho Q: \\ Q \in I \cup K \cup (L \sim \{P_{n,0}, \dots, P_{n,p-1}\}). \end{aligned}$$

In particular, both $f_n(\tau_n(\bar{P}_n)) = \tau_n(\bar{P}_n)$ and $f_n(\sigma_n(\bar{P}_n)) = \sigma_n(\bar{P}_n)$ are derivable from E . We can now conclude that $f_n(\tau_n(\bar{P}_n)) \neq f_n(\sigma_n(\bar{P}_n))$. For otherwise we would have that $\tau_n(\bar{P}_n) = \sigma_n(\bar{P}_n)$ is derivable from E . But since E does not contain any identity

involving any of the constant symbols $P_{n,0}, \dots, P_{n,p-1}$, this would imply that the identity $\tau_n = \sigma_n$ itself is derivable from E , and hence an identity of \mathcal{L}_n since $E \subseteq \Theta_J \mathcal{L}_n$ (see (5) of §3). Thus (2) is established and f_n is well defined for all $n \leq \omega$.

From (3) of §3 and (5), with $n = \omega$, we get $|f_\omega w| < |\rho R_k|$ for every subword w of $\tau_n(\bar{P}_n)$ or $\sigma_n(\bar{P}_n)$, and every k such that $n \leq k < \omega$. Combining this with (2) of §3 we conclude that, in the definition of $f_\omega w$ for each subword of this kind, none of the Cases 4.2, 4.3, 4.6, or 4.7 (with $n = \omega$) can apply for any k except $k < n$. This immediately gives, for all $n < \omega$,

$$f_\omega(\tau_n(\bar{P}_n)) = f_n(\tau_n(\bar{P}_n)) \quad \text{and} \quad f_\omega(\sigma_n(\bar{P}_n)) = f_n(\sigma_n(\bar{P}_n)). \quad (8)$$

This result will be used later.

We now prove that f_ω is a normal-form function for $\mathcal{U}_{e_p} \mathfrak{A}$, i.e., for $\langle A; \Delta_{\mathfrak{A}}; \mathcal{L} \rangle$. For this purpose it suffices to verify the five conditions 2.1(ii), (iii), (i'), (i''), and (i''') with \mathcal{L} , J , $\text{Wd}_J(A)$, $\mathcal{U}_{e_p} \mathfrak{A}$, and $\Delta_{\mathfrak{A}}$ in place of \mathcal{K} , I , $\text{Wd}_I(C)$, \mathfrak{A} and Δ , respectively, and with

$$\Omega = \Theta_I \mathcal{K} \cup \bigcup_{n < \omega} \Gamma_n \cup \{Qx_0, \dots, x_{m-1} = \rho Q : Q \in I \cup K \cup L\} \quad (9)$$

as the set of axioms of \mathcal{L} (see (6) of §3).

For simplicity from now on we omit the subscript on f_ω . By checking each of the nine cases in the definition of f it is easy to see that

$$(f^*w)^{\mathcal{U}_{e_p} \mathfrak{A}} = (fw)^{\mathcal{U}_{e_p} \mathfrak{A}}$$

for every $w \in \text{Wd}_J(A)$. Hence 2.1(ii) holds by a trivial induction argument. To verify the Schreier condition, 2.1(iii), we need the following lemma. We write $v \leq w$ and $v < w$ respectively when v is a subword and a proper subword of w .

$$\begin{aligned} &\text{Let } s \in \text{Wd}_I(A \cup X) \text{ such that } f\chi x = \chi x \text{ for each } x \in X \\ &\text{occurring in } w. \text{ Then } f(s[\chi]) = (gs)[\chi]. \end{aligned} \quad (10)$$

This is proved by induction on $|w|$. The result is obvious if $s \in A \cup X$. Suppose $s \notin A \cup X$. By the induction hypothesis, $f^*(s[\chi]) = (g^*s)[\chi]$. Thus by 4.1, $f(s[\chi]) = (gg^*s)[\chi] = (gs)[\chi]$; the last equality holds since g is a normal-form function.

We now prove 2.1(iii) holds by induction on $|w|$. The result is obvious if $w \in A$, so assume otherwise. We consider the various cases of the definition of fw . If 4.2, 4.3, 4.6, or 4.7 apply we have $v \leq fw < f^*w$. Thus $fv = v$ by the induction hypothesis. In Cases 4.4 and 4.9 we have $v \leq fw = f^*w$. Hence again the desired result follows directly from the induction hypothesis if $v < f^*w$; if $v = f^*w$, then, still again by the induction hypothesis, $f^*v = f^*f^*w = f^*w$, and applying 4.4 or 4.9, whichever is appropriate, $fv = fw = f^*w = v$. Case 4.8 is handled similarly.

Suppose now 4.1 applies, so that $f^*w = t[\chi]$ where $t \in \text{Wd}_I(A \cup X)$ with $t \notin A \cup X$, and $v \leq fw = (gt)[\chi]$. Assume first of all that $v \leq \chi x$ for some $x \in X$ occurring in gt . Then by 2.2(i), either $x = x_0$ or x occurs in t ; in the former case $v = \chi x_0 \in L$ and so $fv = v$ by 4.4. In the latter case $v < f^*w$, so $fv = v$ by the induction hypothesis. We now assume $v = s[\chi]$ for some $s \leq gt$. Since we have

already shown that $f\chi x = \chi x$ for every $x \in X$ occurring in s , we get $fv = (gs)[\chi] = s[\chi] = v$ by (10) and the fact g is a Schreier normal-form function. This finishes Case 4.1; 4.5 is treated similarly. The verification of 2.1(iii) is finished.

Condition 2.1(i') is now easily obtained. We shall actually establish a stronger result that will be needed later on.

$$\text{For any } \pi = \pi(x_0, \dots, x_{m-1}) \in \text{Te}_J \text{ and for all } w_0, \dots, w_{m-1} \in \text{Wd}_J(A), \quad (11)$$

$$f(\pi(w_0, \dots, w_{m-1})) = f(\pi(fw_0, \dots, fw_{m-1})).$$

As usual we prove this by induction on the length of π . If π is a variable, say $\pi = x_i$, then what we have to show is $fw_i = ffw_i$. But this holds by 2.1(iii). Assume $\pi = Q\gamma_0 \dots \gamma_{k-1}$ with $Q \in J$. Then using the induction hypothesis we get

$$\begin{aligned} f^*(\pi(w_0, \dots, w_{m-1})) &= Qf(\gamma_0(w_0, \dots, w_{m-1}), \dots, f(\gamma_{k-1}(w_0, \dots, w_{m-1}))) \\ &= Qf(\gamma_0(fw_0, \dots, fw_{m-1}), \dots, f(\gamma_{k-1}(fw_0, \dots, fw_{m-1}))) \\ &= f^*(\pi(fw_0, \dots, fw_{m-1})). \end{aligned}$$

The desired result now follows at once. Thus 2.1(i') is established.

Let $w = v \in \Delta_{\mathfrak{A}}$. Then by (10) and the fact g is a normal-form function for $\mathfrak{A}[X]$ we have $fw = gw = gv = fv$. Hence 2.1(i'') holds, and it remains only to verify 2.1(i''') where Ω is the set of identities given in (9). Assume $\pi(x_0, \dots, x_{m-1}) = \gamma(x_0, \dots, x_{m-1})$ holds identically in \mathfrak{K} and $w_0, \dots, w_{m-1} \in \text{Wd}_J(A)$. For each $i < m$ let $fw_i = t_i[\chi]$ where $t_i \in \text{Wd}_J(A \cup X)$; observe that $f\chi x = \chi x$ for each x in t_i . Thus

$$\begin{aligned} f(\pi(w_0, \dots, w_{m-1})) &= f(\pi(fw_0, \dots, fw_{m-1})) \quad \text{by (11)} \\ &= f(\pi(t_0, \dots, t_{m-1})[\chi]) \\ &= (g(\pi(t_0, \dots, t_{m-1})))[\chi] \quad \text{by (10)}. \end{aligned}$$

Similarly, $f(\gamma(w_0, \dots, w_{m-1})) = (g(\gamma(t_0, \dots, t_{m-1})))[\chi]$. But $\pi = \gamma \in \Theta_J \mathfrak{K}$ and g is a normal-form function for the presentation $\langle A \cup X; \Delta_{\mathfrak{A}}; \mathfrak{K} \rangle$ of $\mathfrak{A}[X]$. Hence $g(\pi(t_0, \dots, t_{m-1})) = g(\gamma(t_0, \dots, t_{m-1}))$ so that, for every $\pi = \gamma \in \Theta_J \mathfrak{K}$,

$$f(\pi(w_0, \dots, w_{m-1})) = f(\gamma(w_0, \dots, w_{m-1})) \quad \text{for all } w_0, \dots, w_{m-1} \in \text{Wd}_J(A). \quad (12)$$

By (8),

$$f^*(R_n \tau_n(\bar{P}_n) \sigma_n(\bar{P}_n) wv) = R_n f_n(\tau_n(\bar{P}_n)) f_n(\sigma_n(\bar{P}_n)) fwfv.$$

Thus by 4.6, $f(R_n \tau_n(\bar{P}_n) \sigma_n(\bar{P}_n) wv) = fw$. Similarly, by 4.7, $f(R_n wv wv) = fv$. Hence (12) holds for all $\pi = \gamma \in \bigcup_{n < \omega} \Gamma_n$.

Finally, we consider the definitional identities $Qx_0, \dots, x_{m-1} = \rho Q$ for arbitrary $Q \in I \cup K \cup L$. Let $w_0, \dots, w_{m-1} \in \text{Wd}_J(A)$. In view of (11) we assume without loss of generality that $fw_i = w_i$ for all $i < m$. We begin by proving that

$$f(\delta(w_0, \dots, w_{m-1})) = \delta(w_0, \dots, w_{m-1}) \quad \text{for every proper subterm } \delta \text{ of } \rho Q. \quad (13)$$

This is obvious if δ is a variable, so assume otherwise. By the induction hypothesis $f^*(\delta(w_0, \dots, w_{m-1})) = \delta(w_0, \dots, w_{m-1})$. In the definition of $f(\delta(w_0, \dots, w_{m-1}))$ only Cases 4.5–4.9 can apply since the first symbol of δ is \odot . But 4.5–4.8 cannot apply because ρ is nonoverlapping (Definition 2.4). This leaves only 4.9; hence (13) holds. Applying this result we have

$$f^*(\rho Q(w_0, \dots, w_{m-1})) = \rho Q(w_0, \dots, w_{m-1}),$$

and since $fw_i = w_i$ by assumption, we also have $f^*(Qw_0 \dots w_{m-1}) = Qw_0 \dots w_{m-1}$. Thus if, for instance, 4.5 applies in the definition of $f(\rho Q(w_0, \dots, w_{m-1}))$, then 4.1 must apply in defining $f(Qw_0 \dots w_{m-1})$, and these two words must be identical. Similarly, Cases 4.6, 4.7, and 4.8 are paired off respectively with 4.2, 4.3, and 4.4. Therefore $f(\rho Q(w_0, \dots, w_{m-1})) = f(Qw_0 \dots w_{m-1})$ for all $Q \in I \cup K \cup L$.

We have shown that (12) holds for all the defining identities of \mathcal{L} . This gives 2.1(i''') and completes the proof that f is a Schreier normal-form function for $\mathbb{U}_\rho \mathcal{A}$. The function f depends of course on the algebra \mathcal{A} , and in the sequel when we want to indicate this dependence we write $f_{\mathcal{A}}$ for f . Observe that f depends not only on \mathcal{A} but also on the particular normal-form function g for the free extension $\mathcal{A}[X]$ that is used in its construction.

5. Proof of theorem for V the variety of groupoids. The normal-form functions $f_{\mathcal{A}}$ constructed in the preceding section provide a useful representation of the universal envelope functor \mathbb{U}_ρ , both the object and arrow function parts. Let us call a word $w \in \text{Wd}_J(A)$ *reduced*, more precisely $f_{\mathcal{A}}$ -*reduced*, if $f_{\mathcal{A}}w = w$. Imitating what is done in combinatorial group theory we identify the elements of $\mathbb{U}_\rho \mathcal{A}$ with the reduced J -words in the generators A . Then for any $Q \in J$ and all elements w_0, \dots, w_{m-1} of $\mathbb{U}_\rho \mathcal{A}$ we have $Q^{\mathbb{U}_\rho \mathcal{A}}(w_0, \dots, w_{m-1}) = f_{\mathcal{A}}(Qw_0 \dots w_{m-1})$. Consider any $\mathcal{B} \in \mathcal{K}$ and any homomorphism $h: \mathcal{B} \rightarrow \mathcal{A}$. For each word $w \in \text{Wd}_J(B)$ let $\bar{h}w$ be the word in $\text{Wd}_J(A)$ obtained from w by replacing each generator $b \in B$ by hb . It is easy to see that $(\mathbb{U}_\rho h)(w) = f_{\mathcal{A}}\bar{h}w$ for every element w of $\mathbb{U}_\rho \mathcal{B}$.

Suppose now that h is injective. Without loss of generality we assume \mathcal{B} is a subalgebra \mathcal{A} and that h is the identity mapping. Then $\Delta_{\mathcal{B}} \subseteq \Delta_{\mathcal{A}}$. Let g be a Schreier normal-form function for the presentation $\langle A \cup X; \Delta_{\mathcal{A}}; \mathcal{K} \rangle$ of $\mathcal{A}[X]$ that satisfies all four conditions of Lemma 2.2. By Lemma 2.3(ii), g restricted to $\text{Wd}_J(B \cup X)$ becomes a normal-form function for the presentation $\langle B \cup X; \Delta_{\mathcal{B}}; \mathcal{K} \rangle$ of $\mathcal{B}[X]$. Let $f_{\mathcal{A}}$ and $f_{\mathcal{B}}$ be the normal-form functions for $\mathbb{U}_\rho \mathcal{A}$ and $\mathbb{U}_\rho \mathcal{B}$, respectively, that are defined in the preceding section using g and its restriction. It is a routine exercise in checking each of the nine cases of the definitions to prove by induction on the length of words that $f_{\mathcal{A}}w = f_{\mathcal{B}}w$ for all $w \in \text{Wd}_J(B)$. This immediately implies that $\mathbb{U}_\rho \mathcal{B}$ is a subalgebra of $\mathbb{U}_\rho \mathcal{A}$, i.e., $\mathbb{U}_\rho h$ is injective. Therefore \mathbb{U}_ρ as an arrow function preserves injections.

It is even easier to show that the unit of adjunction $\eta_{\mathcal{A}}: \mathcal{A} \rightarrow \mathbb{U}_\rho \mathcal{A}$ is always injective. Indeed, for each $a \in A$ we have $\eta_{\mathcal{A}}a = f_{\mathcal{A}}a = ga = a$, so $\eta_{\mathcal{A}}$ coincides with the identity function on A . Finally, it is clear that \mathbb{U}_ρ is an isomorphism between \mathcal{K} and a subcategory of \mathcal{L} . Hence \mathbb{U}_ρ satisfies all three defining conditions of a polyinjective functor. This proves the first part of Theorem 1.1.

To prove the second part we assume I is finite and $\Theta_I \mathcal{K}$ is decidable. Since \mathcal{L} is equationally complete, $\Theta_J \mathcal{L}$ is decidable if it is recursively enumerable. Hence to prove it is decidable it is sufficient to prove that its axiom set $\Theta_I \mathcal{K} \cup \bigcup_{n < \omega} \Gamma_n \cup \{Qx_0 \dots x_{m-1} = \rho Q : Q \in I \cup K \cup L\}$ is recursive. $\Theta_I \mathcal{K}$ is recursive by assumption, and from the proof of 2.5 given in McNulty [12, Theorem 2.9(v)], it easily follows that the formal definition ρ can be taken so that $\{Qx_0 \dots x_{m-1} = \rho Q : Q \in I \cup K \cup L\}$ is recursive. Thus the problem reduces to showing $\bigcup_{n < \omega} \Gamma_n$ is recursive.

Let \mathfrak{A} be the free algebra of \mathcal{K} with an infinite number of free generators. Since I is finite and $\Theta_I \mathcal{K}$ is decidable, the normal-form function g for $\mathfrak{A}[X]$ can be taken to be recursive; see the remarks following the proof of Lemma 2.2. Let us consider the sequence of normal-form functions f_0, f_1, f_2, \dots constructed by Cases 4.1–4.9 using g . Assume f_k is recursive for all $k \leq n$. f_n is a normal-form function for the presentation $\langle A; \Delta_{\mathfrak{A}}; \mathcal{L}_n \rangle$. Since \mathfrak{A} is free this presentation defines a free algebra of \mathcal{L}_n with an infinite number of free generators. Thus the decidability of $\Theta_J \mathcal{L}_n$ is an immediate consequence of the recursiveness of f_n . From this we get that the first nonidentity $\tau_n = \sigma_n$ of \mathcal{L}_n can be effectively determined, and thus so can the identities of Γ_n . This fact together with the recursiveness of g implies that f_{n+1} is recursive. We conclude that f_n and $\Theta_J \mathcal{L}_n$ are recursive for all n ; moreover, the decision procedure for $\Theta_J \mathcal{L}_n$ is uniform in n . This establishes the recursiveness of $\bigcup_{n < \omega} \Gamma_n$, and as a consequence the decidability of $\Theta_J \mathcal{L}$. The proof of Theorem 1.1 for \mathcal{V} the variety of groupoids is finally finished.

6. Outline of proof of theorem in full generality. It is possible to obtain a result like Theorem 1.1 for the variety of groupoids because there exist formal definitions, namely the nonoverlapping ρ , that give rise to systems of polynomials that are, at least in one important sense, completely independent. But more importantly each of these definitions admits a solution of the word problem for the universal envelope $\text{Ule}_{\rho} \mathfrak{A}$ that completely exposes its structure. Roughly speaking, normal-universal varieties are by definition just those which share this property of groupoids. However groupoids do have one very special property. In §4 the normal form $f_n w$ was constructed from w by recursion on its length. This can be put more suggestively if we think of words in the natural way as ordered, labeled trees: w is transformed into $f_n w$ by moving systematically up w from leaves to root. This straightforward way of constructing $f_n w$ will not work for arbitrary normal-universal varieties. But for groupoids it is a luxury rather than a necessity, and the f_n can be constructed in a more haphazard way which we now describe.

Let w and w' be arbitrary J -words in the generators A , and let $n \leq \omega$ be arbitrary. We say that w is *directly n -reducible* to w' if the latter word can be obtained from the former by replacing an arbitrary subword v of w by another word v' where v and v' are related in exactly the same way $f_n^* w$ and $f_n w$ are related in the statement of any one of the three Cases 4.1, 4.2, or 4.3. From 4.1, for example, we would have $v' = (gt)[x]$ where t is the unique word in $\text{Wd}_I(A \cup X)$ such that $v = t[x]$. The word w is also *directly n -reducible* to w' if w' can be obtained from w by replacing a subword of the form $\rho Q(w_0, \dots, w_{m-1})$ by $Qw_0 \dots w_{m-1}$ for any $Q \in I \cup K \cup L$.

We say that w is *n-reducible* to w' if these two words coincide, or if w' can be obtained from w by an arbitrary finite sequence of direct n -reductions; w is *n-irreducible* if it is not n -reducible to any word but itself. It can be proved by an argument roughly paralleling that of §4 that every I -word in A is n -reducible to an n -irreducible word, and, moreover, that the latter is unique; this unique irreducible n -reduct of w turns out to be the $f_n w$ constructed in §4. The uniqueness of the irreducible reduct of w is a consequence of the fact that the family of direct n -reductions satisfies the so-called *diamond condition*: any two direct n -reducts of a given word are in turn n -reducible to some common word (cf. Bergman [3] or Theorem 3.2 of [14]). If one of the direct n -reducts arises by replacing a subword $R_k f_k(\tau_k(\bar{P}_k)) f_k(\sigma_k(\bar{P}_k)) t s$ by t , and the other by replacing another subword $R_k r r t s$ by s , then these two subwords must occur at different places in the word because we can prove, as we did in §4, that $f_k(\tau_k(\bar{P}_k)) \neq f_k(\sigma_k(\bar{P}_k))$. Hence, loosely speaking, they cannot interfere with one another, and this gives the diamond condition in this case. A similar situation occurs when the two direct n -reducts both arise by replacing a subword of the form $\rho Q(w_0, \dots, w_{m-1})$ by $Q w_0 \dots w_{m-1}$; here of course the nonoverlapping property of ρ plays the essential role.

We turn now to the case \mathcal{V} is an arbitrary normal universal variety. Let M be the language-type of \mathcal{V} . Take \mathcal{K} to be an arbitrary variety of (countable) type I disjoint from M , and let the types K and L be defined as in §3 except that they are both also assumed to be disjoint from M . Let ρ be a *normal universal definition* of the combined language type $I \cup K \cup L$ in \mathcal{V} . (This concept is defined in [14, Definition 3.3]; we shall describe the basic ideas underlying it later on. A *normal universal variety* is one for which such definitions can always be found.) Imitating what was done in §3 we identify \mathcal{V} with its definitional extension to the language type $J = I \cup K \cup L \cup M$ by the definitions ρ . The decreasing sequence $\mathcal{L}_0 \supseteq \mathcal{L}_1 \supseteq \dots \supseteq \mathcal{L}_\omega = \mathcal{L}$ of subvarieties of \mathcal{V} is defined just as before. In particular,

$$\mathcal{L} = \text{Mo}_J \left[\Theta_M \mathcal{V} \cup \Theta_I \mathcal{K} \cup \bigcup_{n < \omega} \Gamma_n \cup \{ Q x_0 \cdots x_{m-1} = \rho Q : Q \in I \cup K \cup L \} \right].$$

Here Γ_n is the local discriminating system defined in (4) of §3 with $\tau_n = \sigma_n$ taken to be the first nonidentity of \mathcal{L}_n . As before \mathcal{L} is clearly seen to be equationally complete provided it is consistent.

The universal envelope $\text{Ue}_\rho \mathcal{A}$ is constructed as before by means of the last member of a transfinite sequence f_n , $n < \omega$, of normal-form functions. The f_n are defined in the manner outlined in the first part of the present section except that we must now take into account the identities of \mathcal{V} . To this end let Y be a denumerable set of free generators disjoint from A where A is the universe of a fixed but arbitrary algebra of \mathcal{K} . Let ψ be a one-one correspondence between Y and the subset of all words in $\text{Wd}_J(A)$ that begin with a symbol in $J \sim M$ ($= I \cup K \cup L$). For any $w \in \text{Wd}_M(A \cup Y)$ we denote by $w[\psi]$ the word in $\text{Wd}_J(A)$ obtained from w by replacing each $y \in Y$ occurring in w by ψy . (ψ will play the same role when we come to deal with the identities of \mathcal{V} that χ did when we dealt in 4.1 and 4.5 with the identities of \mathcal{K} .) There exists a normal-form function h for the free algebra of \mathcal{K} , with free generators $A \cup Y$, that satisfies the obvious analogues of 2.2(i), (ii), (iv), and (1) of §4; the existence of such an h is one

of the defining properties of a normal universal variety. (A more important property will be discussed later.)

Consider any pair of words w and $w' \in \text{Wd}_J(A)$, and let $n \leq \omega$. We define w to be *directly n -reducible* to w' in the new sense if it is directly n -reducible to w' in the sense described in the first part of the section, or (here is where the identities of \mathcal{V} are taken into account) w' is obtained from w by replacing any subword of the form $t[\psi]$ by $(ht)[\psi]$. The notion of w being *n -reducible* to w' (in the new sense) and of w being *n -irreducible* is defined in the obvious way. The main task now consists of proving that each $w \in \text{Wd}_J(A)$ has a unique n -irreducible n -reduct. For if this is done and we take $f_n w$ to be this irreducible reduct, it is an easy matter to prove that f_ω is a normal-form function for $\mathbb{U}_\rho \mathfrak{A}$. Then just as in §5 we can use f_ω to show that \mathbb{U}_ρ is a polyinjective functor from \mathfrak{K} into \mathfrak{L} .

If a simple adjustment is made in the construction of the normal-form function g for $\mathfrak{A}[X]$ used in §4, it can be shown that a direct n -reduct of any word can have no more occurrences of J -symbols than the word itself. Using this fact we can prove without difficulty that every word has at least one n -irreducible reduct. To prove this is unique it suffices to prove that the family of direct n -reductions in the new sense also satisfies the diamond condition. The only real problem here is when two direct n -reducts both arise by replacing a subword of the form $\rho Q(w_0, \dots, w_{m-1})$ by $Qw_0 \dots w_{m-1}$, or one of them is of this type and the other arises by replacing a subword $t[\psi]$ by $(ht)[\psi]$. But that the normal-form function h can be chosen so that in both these situations the diamond condition holds is exactly the defining property of the normal universal definition ρ .

This completes the outline of the proof of Theorem 1.1 for an arbitrary normal universal variety \mathcal{V} . In the case of quasi-groups we take ρ to be one of the *special nonoverlapping* definitions defined on p. 200 of [13]. Then we can define the normal-form functions f_n in basically the same bottom-up manner that works for groupoids, and the fundamental ideas of the two proofs are the same. Some essentially new difficulties do arise, but it is clear from the discussion of [13] how to handle them. See in particular the proof of Theorem 3.9, p. 215.

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